On the twisted convolution product and the Weyl transformation of tempered distributions

J.M. MAILLARD

Laboratoire de Physique-Mathématique
Université de Dijon
B.P. 138
21004 - Dijon Cedex - France

Abstract. *It is well known that the Weyl transformation in a phase space* $\mathbb{R}^{2l}$, *transforms the elements of* $\mathscr{S}(\mathbb{R}^{2l})$ *in trace class operators and the elements of* $L^2(\mathbb{R}^{2l})$ *in the Hilbert-Schmidt operators of the Hilbert space* $L^2(\mathbb{R}^l)$; *this fact leads to a general method of quantization suggested by E. Wigner and J.E. Moyal and developed by M. Flato, A. Lichnerowicz, C. Fronsdal, D. Sternheimer and F. Bayen for an arbitrary symplectic manifold, known under the name of star-product method. In this context, it is important to study the Weyl transforms of the tempered distributions on the phase space and that of the star-exponentials which give the spectrum in this process of quantization.*
We analyse here the relations between the star-product, the twisted convolution product and the Weyl transformation of tempered distributions. We introduce symplectic differential operators *which permit us to study the structure of the space* $\mathcal{O}'_\lambda$, $\lambda \neq 0$, *(similar to the space* $\mathcal{O}'_C$*) of the left (twisted) convolution operators of* $\mathscr{S}(\mathbb{R}^{2l})$ *which permit to define the twisted convolution product in the space* $\mathscr{S}'(\mathbb{R}^{2l})$, *and the structures of the admissible symbols for the Weyl transformation (i.e. the domain of the Weyl transformation). We prove that the bounded operators of* $L^2(\mathbb{R}^l)$ *are exactly the Weyl transforms of the bounded (twisted) convolution operators of* $L^2(\mathbb{R}^{2l})$. *We give an expression of the integral formula of the star product in terms of twisted convolution products which is valid in the most general case. The unitary representations of the Heisenberg group play an important role here.*

Key-Words: Symplectic Fourier transforms, star products.
1980 Mathematics Subject Classification: 81 C 99, 43 A 32.

INTRODUCTION

The Weyl transformation is a one-to-one mapping $f \to \mathrm{op}(f)$ of a large family of functions or distributions (including polynomials) on the phase space $\mathbb{R}^{2l} = \{q_j, p_j\}$, $1 \leqslant j \leqslant l$ of a non-relativistic system with l degrees of freedom onto a large class of operators of the Hilbert space $\mathbb{H} = L^2(\mathbb{R}^l)$, including those which are Hilbert-Schmidt (The Hilbert-Schmidt operators are the Weyl transforms of the square integrable functions on the phase space), such that:

$$\mathrm{op}(1) = 1, \quad \mathrm{op}(q_j) = q_j, \quad \mathrm{op}(p_j) = \frac{\hbar}{i} \frac{\partial}{\partial q_j} .$$

The Weyl transform is a usual quantization process.

The star product $f \circ g$ of two functions or distributions f and g on the phase space is the symbol of the operator $\mathrm{op}(f)\,\mathrm{op}(g)$ (when defined). As shown by J.E. Moyal, the commutator for the star product appears (at least formally) as an asymptotic expansion which is a deformation with parameter $i(\hbar/2)$ of the Poisson bracket on the phase space. This fact leads to consider quantum mechanic as a theory on the space of functions or distributions over the phase space. The equations of motion in the Heisenberg picture are then obtained from the classical equations of motion by using that deformation of the Poisson bracket (see [1]). In this context, M. Flato, A. Lichnerowicz, C. Fronsdal, D. Sternheimer and F. Bayen have constructed a new quantization process on an arbitrary symplectic manifold by considering deformations of the ordinary product and of the Poisson bracket of the symplectic structure. They have computed the spectrum of some Hamiltonians by considering a Fourier-Dirichlet expansion of their star-exponentials.

In this paper, we study properties of the Weyl transformation which relates the usual quantization process and the star-product method in the case of the phase space $\mathbb{R}^{2l}$, and of the twisted convolution product noted $\underset{\lambda}{*}$, which is the Fourier transform of the star product. Many authors have been working on these questions. Nevertheless, many problems were not solved, in particular the determination of the domain of the Weyl transformation (what we call the space of the admissible symbols, which is not the space $\mathcal{O}'_C$, as we show in section 7), the characterization of the symbols of the bounded operators on the Hilbert space $\mathbb{H}$, the determination of the structure of the left convolution operators of $\mathcal{S}(\mathbb{R}^{2l})$ (introduced by M.A. Antonets in order to define the twisted convolution product in the space $\mathcal{S}'(\mathbb{R}^{2l})$) and the structure of the admissible symbols.

In section 1, we recall the usual definition of the Weyl transformation.

In section 2, we show that the Weyl transformation is in fact a one-to-one mapping from the space $\mathscr{S}'(\mathbb{R}^{2l})$ onto the space $\mathscr{L}(\mathscr{S}(\mathbb{R}^l), \mathscr{S}'(\mathbb{R}^l))$ of continuous linear maps of $\mathscr{S}(\mathbb{R}^l)$ into $\mathscr{S}'(\mathbb{R}^l)$. We call admissible symbols the tempered distributions T of $\mathscr{S}'(\mathbb{R}^{2l})$ such that $\mathrm{op}(T)(\mathscr{S}(\mathbb{R}^l)) \subset \mathbb{H} = L^2(\mathbb{R}^l)$, i.e. those that give operators in the Hilbert space $\mathbb{H}$.

In section 3, we introduce a new kind of differential operators that we call *symplectic differential operators*. These differential operators will permit us to treat the twisted convolution product in a manner similar to that used for the ordinary convolution product, and will permit us to determine the structures of the left convolution operators of $\mathscr{S}(\mathbb{R}^{2l})$ and of the admissible symbols of the Weyl transformation.

In section 4, we study the Fourier transforms of twisted convolution products and we give an integral formula for the star product in terms of twisted convolution products, which is valid in the most general case. We see in particular in this section, that the Fourier transform of a twisted convolution product for $\lambda \neq 0$ is no more an ordinary product of functions, but a twisted convolution product for $1/\lambda$. The notion of space $\mathcal{O}_M$ is thus lost in this case.

In section 5, we study the structures of the left convolution operators of $\mathscr{S}(\mathbb{R}^{2l})$. We obtain a theorem of structure similar to the theorem of structure of the space $\mathcal{O}'_C$, where the ordinary differential operators are replaced by the symplectic ones.

In section 6, we study the Weyl transforms of the left convolution operators of $\mathscr{S}(\mathbb{R}^{2l})$. We give a characterization of the admissible symbols in terms of twisted convolution products and the structure of these symbols. The admissible symbols are in fact square integrable functions on the phase space and finite sums of symplectic derivatives of square integrable functions. We introduce the notion of bounded (twisted) convolution operators of the space $L^2(\mathbb{R}^{2l})$ and we show that the bounded operators of the Hilbert space $\mathbb{H} = L^2(\mathbb{R}^l)$ are the Weyl transforms of these bounded convolution operators.

In section 7, we give some practical applications of these results. We show in particular that the Weyl transforms of the star exponentials of the homogeneous polynomials of degree two on the phase space give rise to one-parameter groups of unitary operators.

1. GENERAL RESULTS AND NOTATIONS

Let $E = \mathbb{R}^{2l}$ be the phase space of a nonrelativistic quantum system with l degrees of freedom, whose points are denoted by: $x = (q_j, p_j)$, $1 \leqslant j \leqslant l$ or $x = q + p$. Let $\omega = \sum_{j=1}^{l} \mathrm{d}q_j \wedge \mathrm{d}p_j$ be a symplectic form on E and let $\mathrm{d}\mu(x) =$

$= \dfrac{1}{(2\pi)^l}\, \mathrm{d}q \cdot \mathrm{d}p$ be the measure associated with the symplectic form ω. Let us consider the Weyl group associated with E (see [2]), $W = E \times \mathbb{R}$ endowed with the composition law:

$$(x_1, s_1)(x_2, s_2) = (x_1 + x_2, s_1 + s_2 + 1/2\ \omega(x_1, x_2))$$

Let us note that this group is isomorphic to the Heisenberg group. The Lie algebra w of this group is isomorphic to the space $E \times \mathbb{R}$ endowed with the Lie bracket:

$$[(x_1, s_1), (x_2, s_2)] = (0, \omega(x_1, x_2))$$

and we have: $\exp(x, s) = (x, s)$.

The center of the Weyl group is the set of elements $(0, s)$; and with every nonzero real number $\hbar$ is associated an equivalence class of irreducible unitary representations of the Weyl group, satisfying the formula:

$$\pi(0, s) = \exp(-is/\hbar)\, I.$$

Let us consider the symplectic Fourier transformation in the space $\mathscr{S}(E)$ of infinitely differentiable functions, rapidly decreasing at infinity (see [5]), defined by:

$$\mathscr{F}\varphi(x) = \hat{\varphi}(x) = \int e^{i\omega(x,y)} \cdot \varphi(y)\, \mathrm{d}\mu(y).$$

Then $\mathscr{F}^{-1} = \mathscr{F}$ and this Fourier transformation can be extended to an isometry of the space $L^2(E, \mu)$ of square-integrable functions with respect to the measure μ onto itself. The symplectic Fourier transform of the tempered distribution T of $\mathscr{S}'(E)$ is defined by: $\langle \mathscr{F}T, \varphi\rangle = \langle T, \mathscr{F}\check{\varphi}\rangle$, where $\check{\varphi}(x) = \varphi(-x)$. Let us note that: $(\mathscr{F}\varphi)^{\check{}} = \mathscr{F}(\check{\varphi})$.

REMARK. *To every locally integrable function $f(x)$ in E is associated the measure $f(x) \cdot \mathrm{d}\mu(x)$.*

The Weyl transforms (see [1] and [2]) of the functions φ of $\mathscr{S}(E)$ are bounded operators of $\mathbb{H}$, defined by $\mathrm{op}(\varphi) = \int \hat{\varphi}(x)\, \pi(-\hbar x, 0)\, \mathrm{d}\mu(x)$, where π is an irreducible unitary representation of W, associated with $\hbar \neq 0$. The star product $\varphi \circ \psi$ of the functions φ and ψ of $\mathscr{S}(E)$ is defined by (see [1]): $\mathrm{op}(\varphi \circ \psi) = \mathrm{op}(\varphi) \cdot \mathrm{op}(\psi)$ and we have: $\mathscr{F}(\varphi \circ \psi) = \hat{\varphi} \underset{\hbar/2}{*} \hat{\psi}$, where $\underset{\hbar/2}{*}$ denotes the twisted convolution product associated with $\dfrac{\hbar}{2}$.

Let us note that these notations may be different from those used in the references quoted in this paper. The usual notations in distribution theory used in this paper are those of [12]. Unless explicitly mentioned, the signs $\mathcal{F}$ and $\hat{\varphi}$ will denote through this paper the symplectic Fourier transformation.

2. WEYL TRANSFORMS OF TEMPERED DISTRIBUTIONS

Let π be an irreducible unitary representation of the Weyl group, belonging to the equivalence class associated with $\hbar \neq 0$. Let us denote by $\mathbb{H}$ the Hilbert space of this representation and by $\mathbb{H}_\infty$ the space of differentiable vectors. Let us set: $\chi_{u,v}(x) = (\pi(-\hbar x, 0)\, u \mid v)$ for u and v in $\mathbb{H}$, where $(\,|\,)$ denotes the scalar product in $\mathbb{H}$. Let us denote by $\|u\|$ the norm of the vector u of $\mathbb{H}$. We have the lemma:

LEMMA 1.

1) *For all vectors u and v of $\mathbb{H}$, the coefficient $\chi_{u,v}$ belongs to $L^2(E)$ and satisfies the formula:*

$$\int |\chi_{u,v}(x)|^2 \,\mathrm{d}\mu(x) = |\hbar|^{-l} \|u\|^2 \|v\|^2. \tag{1}$$

2) *For all vectors u and v of $\mathbb{H}_\infty$, the coefficient $\chi_{u,v}$ belongs to $\mathcal{S}(E)$.*

Let us consider on $\mathbb{H}_\infty$ the topology defined by the family of seminorms (see [7]):

$$\rho_{\mathcal{U}}(u) = \| \mathrm{d}\pi(\mathcal{U})\, u \| \tag{2}$$

for all $\mathcal{U}$ belonging to the universal enveloping algebra of w, where $\mathrm{d}\pi$ is the differential of π. Then, $\mathbb{H}_\infty$ is isomorphic to $\mathcal{S}(\mathbb{R}^l)$ and $u, \bar{v} \to \chi_{u,v}$ is a continuous bilinear map of $\mathbb{H}_\infty \times \mathbb{H}_\infty$ into $\mathcal{S}(E)$.

We can now define the Weyl transform of a tempered distribution $T \in \mathcal{S}'(E)$ by setting for all $u, v \in \mathbb{H}_\infty$:

$$(\mathrm{op}\,(T)\, u \mid v) = \langle \hat{T}, \chi_{u,v} \rangle. \tag{3}$$

Let us note that $\mathbb{H}_\infty$ being isomorphic to $\mathcal{S}(\mathbb{R}^l)$ (for the structures of topological vector spaces), its dual space $\mathbb{H}'_\infty$ is isomorphic to $\mathcal{S}'(\mathbb{R}^l)$ and we set for $f \in \mathcal{S}'(\mathbb{R}^l)$ and $u \in \mathcal{S}(\mathbb{R}^l)$: $(f \mid u) = \langle f, \bar{u} \rangle$. We have the proposition:

PROPOSITION 1. *The Weyl transformation is a one to one mapping ot the space $\mathcal{S}'(E)$ of tempered distributions in E onto the space $\mathcal{L}(\mathbb{H}_\infty, \mathbb{H}'_\infty)$ of continuous* *linear maps of* $\mathbb{H}_\infty$ *into its dual* $\mathbb{H}'_\infty$.

DEFINITION 1. *A tempered distribution T of $\mathscr{S}'(E)$ is called an admissible symbol for the Weyl transformation if* $\mathrm{op}\,(T)(\mathbb{H}_\infty) \subset \mathbb{H}$.

Let us note that $\mathbb{H}$ being isomorphic to $L^2(\mathbb{R}^l)$, is a subspace of $\mathbb{H}'_\infty$. By using the closed graph theorem (see [13], chapter 17), we see that, if T is an admissible symbol, then $\mathrm{op}\,(T)$ is a continuous linear map of $\mathbb{H}_\infty$ into $\mathbb{H}$. So, we have the proposition:

PROPOSITION 2. *The Weyl transformation establishes a one to one mapping of the space of admissible symbols onto the space $\mathscr{L}(\mathbb{H}_\infty, \mathbb{H})$ of continuous linear maps of $\mathbb{H}_\infty$ into $\mathbb{H}$.*

In particular, every bounded operator of $\mathbb{H}$ is the Weyl transform of only one admissible symbol.

Proof. Let us now prove lemma 1 and proposition 1. Since the coefficients of two unitary equivalent representations are equal and since for such two representations, there is a one to one correspondence between the spaces of differentiable vectors of these representations which is a topological isomorphism for the topologies induced by the seminorms (2), we can consider the representation of W on the space $L^2(\mathbb{R}^l)$ equipped with the scalar product: $(u \mid v) = \int u(q')\overline{v(q')}\,\mathrm{d}q'$, ($\mathrm{d}q'$ is the Lebesgue's measure on $\mathbb{R}^l$), defined by:

$$\pi(x, s)u(q') = \exp\left[-\frac{i}{\hbar}\left(s + \frac{qp}{2} + pq'\right)\right]u(q' + q) \tag{4}$$

where $x = q + p$, $qp = \sum_{j=1}^{l} q_j p_j$. The space of the differentiable vectors of this representation is $\mathscr{S}(\mathbb{R}^l)$ (see [7]) and by using the remark following the theorem XVIII (chapter VI, p. 190) of [12], it is easy to prove that the topology defined by the family of seminorms (2) is identical with the usual topology of $\mathscr{S}(\mathbb{R}^l)$. Then, for u and $v \in L^2(\mathbb{R}^l)$ we have:

$$\chi_{u,v}(x) = \exp\left(-i\,\frac{\hbar}{2}\,qp\right)\int e^{ipq'} \cdot u(q' - \hbar q)\,\overline{v(q')}\,\mathrm{d}q'.$$

From the properties of the partial Fourier transformation (with respect to q'), we deduce formula (1). Furthermore, let us consider the topological isomorphism $\mathscr{T}$ of $\mathscr{S}(E)$ onto itself defined by:

(5) $$\mathscr{T}\xi(q,p) = \exp\left(-i\,\frac{\hbar}{2}\,qp\right)\int e^{ipq'}\cdot\xi(q'-\hbar q, q')\,dq'$$

we have $\chi_{u,v} = \mathscr{T}(u \otimes \bar{v})$. The proof of lemma 1 is then complete.

Furthermore, it is clear that $u, \bar{v} \to \chi_{u,v}$ is a continuous bilinear map of $\mathscr{S}(\mathbb{R}^l) \times \mathscr{S}(\mathbb{R}^l)$ into $\mathscr{S}(E)$. Proposition 1 is easily deduced from the formula $\langle \hat{T}, \chi_{u,v}\rangle = \langle {}^t\mathscr{T}(\hat{T}), u \otimes \bar{v}\rangle$ where ${}^t\mathscr{T}$ is the transpose of $\mathscr{T}$, by noticing that $\mathscr{S}(\mathbb{R}^l)$ is a nuclear Frechet space and by using the proposition 50-7 and the corollary of theorem 51-6 of [13]. Let us note that ${}^t\mathscr{T}(\hat{T})$ is the kernel of the operator op (T). ■

We deduce from the proposition 50-4 of [13] the topological isomorphism:

$$\mathscr{S}'(\mathbb{R}^l)\,\hat{\otimes}\,L^2(\mathbb{R}^l) = \mathscr{L}(\mathscr{S}(\mathbb{R}^l), L^2(\mathbb{R}^l)).$$

Then, we have the corollary:

COROLLARY 1. *The admissible symbols for the Weyl transformation are the tempered distributions in E whose kernels belong to* $\mathscr{S}'(\mathbb{R}^l)\,\hat{\otimes}\,L^2(\mathbb{R}^l)$.

3. TWISTED CONVOLUTION PRODUCT

The twisted convolution of two functions f and g in E, defined for every real number λ by:

$$f \underset{\lambda}{*} g(x) = \int e^{-i\lambda\omega(x,y)}\cdot f(x-y)\,g(y)\,d\mu(y)$$

was introduced in [4] Let us note for every index $J = (j_1, \ldots, j_r) : |J| = r$ and

$$D_J = \frac{\partial^r}{\partial x_{j_1} \ldots \partial x_{j_r}}.$$

It will be very useful to introduce the symplectic differential operators associated with the real number λ and defined by:

$$L_j^\lambda \varphi(x) = \frac{\partial\varphi}{\partial x_j} + i\lambda\omega\,(e_j, x)\,\varphi(x)$$

and $L_J^\lambda = L_{j_1}^\lambda \bullet \ldots \bullet L_{j_r}^\lambda$ where (x_j) is a coordinate system with respect to a basis (e_j) in E. Let us note that the differential operators L_j^λ do not commute in general.

If $T \in \mathscr{D}'(E)$ and $\varphi \in \mathscr{D}(E)$ (resp. $T \in \mathscr{S}'(E)$ and $\varphi \in \mathscr{S}(E)$), we have:

$$\langle L_J^\lambda T, \varphi\rangle = (-1)^{|J|} \langle T, L_{\tilde{J}}^{-\lambda} \varphi\rangle$$

where $\tilde{J} = (j_r, \ldots, j_1)$ if $J = (j_1, \ldots, j_r)$.

These symplectic differential operators will permit us to treat the twisted convolution product in a manner similar to that used for the ordinary convolution product. It seems that these differential operators should play an important role in generalizations to other groups.

LEMMA 2. *For every real number* $\lambda \neq 0$, *we have:*

(6-a) $$L_J^\lambda = D_J + \sum_{|S|<|J|} a_S(x)\, D_S$$

(6-b) $$D_J = L_J^\lambda + \sum_{|S|<|J|} L_S^\lambda\, b_S(x)$$

(6-c) $$D_J = L_J^\lambda + \sum_{|S|<|J|} C_S(x)\, L_S^\lambda$$

where $a_S(x)$, $b_S(x)$ *and* $C_S(x)$ *are polynomials of degree not greater than* $|J|$.

Proof. Indeed, the first formula is easily obtained by induction. To obtain the second and third formulas, let us first note that if $\alpha \in \xi(E)$ and $T \in \mathscr{D}'(E)$, we have:

$$L_j^\lambda(\alpha T) = \alpha L_j^\lambda T + \frac{\partial \alpha}{\partial x_j} \cdot T$$

and the formulas (6-b) and (6-c) are then obtained by induction. ■

It is easy to show that, for every real number λ, an infinitely differentiable function φ in E belongs to $\mathscr{S}(E)$ if and only if, for every nonnegative integer r and every index J:

(7) $$\rho_{r,J}^\lambda(\varphi) = \sup_{x \in E} (1 + \|x\|^2)^r \, |L_J^\lambda \varphi(x)| < +\infty$$

and it is easy to prove by using lemma 2, that the topology defined on $\mathscr{S}(E)$ by these seminorms is identical with the usual topology on $\mathscr{S}(E)$.

The twisted convolution (with respect to λ) of two distributions S and T, one of them at least having a compact support, is defined for $\varphi \in \mathscr{D}(E)$ by (see [5]):

(8) $$\langle S \underset{\lambda}{*} T, \varphi\rangle = \langle S(x), \theta(x)\rangle,$$

with:

$$\theta(x) = \langle T(y), e^{-i\lambda\omega(x,y)} \cdot \varphi(x+y)\rangle.$$

To define this twisted convolution product in $\mathscr{S}'(E)$, it is sufficient that the mapping $\varphi \to \theta$ be continuous of $\mathscr{S}(E)$ into itself. This property is easily verified if $T = f \in \mathscr{S}(E)$, since we have in this case: $\theta = \check{f} \underset{\lambda}{*} \varphi$ and since the twisted convolution of two functions of $\mathscr{S}(E)$ is continuous. Then, if $S \in \mathscr{S}'(E)$ and $f \in \mathscr{S}(E)$, the twisted convolution $S \underset{\lambda}{*} f$ is defined and we have the proposition:

PROPOSITION 3. *If S belongs to $\mathscr{S}'(E)$ and f belongs to $\mathscr{S}(E)$, then $S \underset{\lambda}{*} f$ belongs to $\mathcal{O}_M(E)$ and is given by:*

(9-a) $$S \underset{\lambda}{*} f(x) = \langle S(y), e^{i\lambda\omega(x,y)} \cdot f(x-y)\rangle$$

and satisfies the formulas:

(9-b) $$L_J^{\lambda}(S \underset{\lambda}{*} f) = (L_J^{\lambda} S) \underset{\lambda}{*} f$$

(9-c) $$L_J^{-\lambda}(S \underset{\lambda}{*} f) = S \underset{\lambda}{*} L_J^{-\lambda} f.$$

Proof. Indeed, we have:

$$\langle S \underset{\lambda}{*} f, \varphi\rangle = \langle S(x), \int e^{-i\lambda\omega(x,y)} \cdot f(y-x)\, \varphi(y)\, \mathrm{d}\mu(y)\rangle =$$

$$= \int \varphi(y)\, \langle S(x), e^{-i\lambda\omega(x,y)} \cdot f(y-x)\rangle\, \mathrm{d}\mu(y).$$

To justify the preceding inversion under the integral sign, it suffices to note that:

$$L_{x_j}^{\lambda}(e^{-i\lambda\omega(x,y)} \cdot f(y-x)) = -e^{-i\lambda\omega(x,y)} \cdot (L_j^{\lambda} f)(y-x)$$

and that in view of the theorem VI (chapter VII) of [12] and in view of Lemma 2, formula (6-b), every tempered distribution can be written in the form: $S = \sum_J L_J^{-\lambda}((1+\|x\|^2)^k\, h_J(x))$ (finite sum), where the functions h_J are continuous and bounded. We then obtain:

$$\langle S \underset{\lambda}{*} f, \varphi\rangle = \sum_J \int (1+\|x\|^2)^k h_J(x)\,\mathrm{d}\mu(x) \int e^{-i\lambda\omega(x,y)}\cdot(L_J^\lambda f)(y-x)\varphi(y)\,\mathrm{d}\mu(y).$$

So as to inverse the order of integration, let us note that we have for all integers $r, s \geqslant 0$:

$$|(1+\|x\|^2)^k h_J(x)((L_J^\lambda f)(y-x))\,\varphi(y)| \leqslant K_{r,s}\,\frac{(1+\|x\|^2)^k}{(1+\|y-x\|^2)^r(1+\|y\|^2)^s}$$

and then, let us use the following lemma (see [3], lemma 2.3.2, p. 113):

LEMMA 3. *For all vectors x and y of E, we have:*

$$(1+\|x+y\|^2)^{\pm 1} < 2(1+\|x\|^2)^{\pm 1}(1+\|y\|^2).$$

We deduce from this lemma that the preceding quantity under the integral sign is majorized by $2^r k_{rs}(1+\|x\|^2)^{k-r}(1+\|y\|^2)^{r-s}$, which is summable for sufficiently large values of r and $s-r$. A similar computation shows that $S \underset{\lambda}{*} f \in \mathcal{O}_M(E)$.

We obtain the formula (9-c) by writing:

$$L_{x_j}^{-\lambda}(S \underset{\lambda}{*} f(x)) = \langle S(y),\, e^{i\lambda\omega(x,y)}\cdot(L_j^{-\lambda} f)(x-y)\rangle.$$

To obtain the formula (9-b), let us first note that:

$$L_{x_j}^{\lambda}(e^{i\lambda\omega(x,y)}\cdot f(x-y)) = -L_{y_j}^{-\lambda}(e^{i\lambda\omega(x,y)}\cdot f(x-y))$$

and then, let us write:

$$L_j^\lambda(S \underset{\lambda}{*} f)(x) = -\langle S(y), L_{y_j}^{-\lambda}(e^{i\lambda\omega(x,y)}\cdot f(x-y))\rangle =$$

$$= \langle L_{y_j}^{\lambda} S(y), e^{i\lambda\omega(x,y)}\cdot f(x-y)\rangle. \qquad \blacksquare$$

PROPOSITION 4. *The twisted convolution* $S \underset{\lambda}{*} f$ *is hypocontinuous of* $\mathcal{S}'(E)\times\mathcal{S}(E)$ *into* $\mathcal{S}'(E)$.

Proof. This proposition follows from the formula $\langle S \underset{\lambda}{*} f, \varphi\rangle = \langle S, \check{f} \underset{\lambda}{*} \varphi\rangle$, by using the continuity of the twisted convolution product in $\mathcal{S}'(E)$ and by using the fact that every bounded set of $\mathcal{S}'(E)$ is equicontinuous. ■

Following [6], let us introduce the space $\mathcal{O}'_\lambda(E)$ of left convolution operators of $\mathcal{S}(E)$.

DEFINITION 2. *Let λ be a real number. The space $\mathcal{O}'_\lambda(E)$ of left convolution operators of $\mathcal{S}(E)$ is the set of the tempered distributions T in E such that $T \underset{\lambda}{*} \varphi$ belongs to $\mathcal{S}(E)$ for every function φ of $\mathcal{S}(E)$.*

Let us set for $T \in \mathcal{S}'(E)$, $\varphi \in \mathcal{S}(E)$ and $h \in E$:

$$\tau_h \varphi(x) = \varphi(x-h), \langle \tau_h T, \varphi \rangle = \langle T, \tau_{-h} \varphi \rangle.$$

Then we have the proposition:

PROPOSITION 5.

1) *A tempered distribution T in E belongs to $\mathcal{O}'_\lambda(E)$ if and only if, for every nonnegative integer r, the set of the distributions T_h, $h \in E$, where:*

$$T_h(x) = (1 + \|h\|^2)^r \, \tau_{-h} \, [e^{i\lambda\omega(h,x)} \cdot T(x)]$$

is a bounded set of $\mathcal{S}'(E)$.

2) *For every distribution T of $\mathcal{O}'_\lambda(E)$, the map $\varphi \to T \underset{\lambda}{*} \varphi$ is continuous of $\mathcal{S}(E)$ into itself.*

Proof. Indeed, if $T \in \mathcal{O}'_\lambda(E)$, we have for every integer $r \geqslant 0$ and every function φ of $\mathcal{S}(E)$:

$$\sup_{x \in E} (1 + \|x\|^2)^r \, | T \underset{\lambda}{*} \varphi(x) | < +\infty$$

It follows that for every function $\varphi \in \mathcal{S}(E)$, we have:

$$\sup_{x \in E} (1 + \|x\|^2)^r \, | \langle \tau_{-x} (e^{i\lambda\omega(x,y)} \cdot T(y)), \check{\varphi}(y) \rangle | < +\infty \, .$$

It follows therefore that the set of the distributions T_x, $x \in E$, being weakly bounded in $\mathcal{S}'(E)$, is strongly bounded in $\mathcal{S}'(E)$.

Conversely, let us suppose that the set of the tempered distributions T_x, $x \in E$ is a bounded set of $\mathcal{S}'(E)$, then by replacing φ by $L_J^{-\lambda} \varphi$ in the preceding formula and by using the formula (9-c) of proposition 3, we obtain $\rho_{r,J}^{-\lambda}(T \underset{\lambda}{*} \varphi) < +\infty$, where $\rho_{r,J}^{-\lambda}$ is defined by formula (7). It follows that $T \in \mathcal{O}'_\lambda(E)$. Futhermore, the set of the distributions T_x, $x \in E$ being equicontinuous in $\mathcal{S}'(E)$, we see that $\rho_{r,J}^{-\lambda}(\varphi)$ tends to zero if φ tends to zero in $\mathcal{S}(E)$. Proposition 5 is then proved. ∎

REMARK. *$T \in \mathcal{O}'_\lambda(E)$ if and only if $\check{T} \in \mathcal{O}'_\lambda(E)$ and we have: $(T \underset{\lambda}{*} \varphi)\check{} = \check{T} \underset{\lambda}{*} \check{\varphi}$.*

If $T \in \mathcal{O}'_\lambda(E)$ and $S \in \mathcal{S}'(E)$, then $S \underset{\lambda}{*} T$ is defined. If one at least of these distributions have a compact support, the expressions $S(x)$ and $T(y)$ can be exchanged in the right hand side of formula (8). so, in the general case, we are led to introduce the space of right convolution operators of $\mathcal{S}(E)$ (with respect to λ). Let us note that $\varphi \underset{\lambda}{*} S = S \underset{-\lambda}{*} \varphi$; then it follows that the right convolution operators of $\mathcal{S}(E)$ for λ, are nothing else but the elements of $\mathcal{O}'_{-\lambda}(E)$. If $S \in \mathcal{O}'_{-\lambda}(E)$ and $T \in \mathcal{S}'(E)$, then $S \underset{\lambda}{*} T$ is still defined and we have:

$$(10) \qquad \langle S \underset{\lambda}{*} T, \varphi\rangle = \langle T(y), \langle S(x), e^{-i\lambda\omega(x,y)} \cdot \varphi(x+y)\rangle\rangle.$$

So, we have the theorem:

THEOREM 1. *Let S and T be two tempered distributions in E, such that S belongs to $\mathcal{O}'_{-\lambda}(E)$ or T belongs to $\mathcal{O}'_{\lambda}(E)$, then $S \underset{\lambda}{*} T$ is defined and satisfies the formulas:*

1) $\langle S \underset{\lambda}{*} T, \varphi\rangle = \langle S, \check{T} \underset{\lambda}{*} \varphi\rangle = \langle \check{S}, T \underset{\lambda}{*} \check{\varphi}\rangle$ *if* $T \in \mathcal{O}'_{\lambda}(E)$
2) $\langle S \underset{\lambda}{*} T, \varphi\rangle = \langle T, \check{S} \underset{-\lambda}{*} \varphi\rangle = \langle \check{T}, S \underset{-\lambda}{*} \check{\varphi}\rangle$ *if* $S \in \mathcal{O}'_{-\lambda}(E)$

The right hand sides of these formulas being equal if simultaneously S belongs to $\mathcal{O}'_{-\lambda}(E)$ and T belongs to $\mathcal{O}'_{\lambda}(E)$.

Proof. Indeed, it suffices to prove that the right hand sides of formulas (8) and (10) are equal if simultaneously $S \in \mathcal{O}'_{-\lambda}(E)$ and $T \in \mathcal{O}'_{\lambda}(E)$. This fact is easily shown if one of these distributions belongs to $\mathcal{S}(E)$, it suffices in this case to consider the structure of the tempered distributions (see [12]). In the general case, we will use the following lemmas:

LEMMA 4. *Existence of an approximation of the identity in the algebra $(\mathcal{S}(E), \underset{\lambda}{*})$.*

Let (φ_n) be a sequence of functions of $\mathcal{D}(E)$ such that, for every nonnegative integer n : $\varphi_n \geqslant 0$, $\varphi_n(x) = 0$ if $\|x\| > 1/n$ and $\int \varphi_n(x)\, \mathrm{d}\mu(x) = 1$. Then, for every function φ of $\mathcal{S}(E)$, the sequence $(\varphi_n \underset{\lambda}{*} \varphi)$ converges to φ in $\mathcal{S}(E)$.

This lemma is proved in a manner similar to that used in the case of the usual convolution product, by using formula (9-c).

LEMMA 5. *Let (φ_n) be a sequence of functions of $\mathcal{D}(E)$ satisfying the hypotheses of lemma 4, then for every tempered distribution S and every real number λ, the sequence $(S \underset{\lambda}{*} \varphi_n)$ converges to S in $\mathcal{S}'(E)$.*

The proof of this lemma is an immediate consequence of lemma 4 and of the fact that every weakly convergent sequence in $\mathcal{S}'(E)$ is strongly convergent.

LEMMA 6. *Let S be a tempered distribution in E and let φ and ψ be two functions of $\mathcal{S}(E)$, then for every real number λ, we have:*

$$(S \underset{\lambda}{*} \varphi) \underset{\lambda}{*} \psi = S \underset{\lambda}{*} (\varphi \underset{\lambda}{*} \psi).$$

The proof of this lemma is obvious.

Let us now return to the proof of theorem 1. Let $S \in \mathcal{O}'_{-\lambda}(E)$ and $T \in \mathcal{O}'_{\lambda}(E)$ and let (φ_n) be a sequence of functions of $\mathcal{D}(E)$ satisfying the hypotheses of lemma 4. Then according to lemma 5, $\langle S \underset{-\lambda}{*} \varphi_n, \check{T} \underset{\lambda}{*} \varphi\rangle$ tends to $\langle S, \check{T} \underset{\lambda}{*} \varphi\rangle$ as n tends to infinity. Furthermore, this first expression is equal to $\langle \check{T}, (S \underset{-\lambda}{*} \varphi_n) \underset{-\lambda}{*} \check{\varphi}\rangle$, since $S \underset{-\lambda}{*} \varphi_n \in \mathcal{S}(E)$; then according to lemmas 4 and 6 and to proposition 5,2), this last expression tends to $\langle \check{T}, S \underset{-\lambda}{*} \check{\varphi}\rangle$. Theorem 1 is then proved. ∎

PROPOSITION 6. *If T belongs to $\mathcal{O}'_{\lambda}(E)$, then $L_j^{\pm\lambda} T$ belongs to $\mathcal{O}'_{\lambda}(E)$; and if T belongs to $\mathcal{O}'_{\lambda}(E)$ or if S belongs to $\mathcal{O}'_{-\lambda}(E)$, we have:*

1) $L_j^{\lambda}(S \underset{\lambda}{*} T) = (L_j^{\lambda} S) \underset{\lambda}{*} T$
2) $L_j^{-\lambda}(S \underset{\lambda}{*} T) = S \underset{\lambda}{*} L_j^{-\lambda} T$
3) $(L_j^{-\lambda} S) \underset{\lambda}{*} T = S \underset{\lambda}{*} L_j^{\lambda} T.$

This proposition follows from proposition 3 and from the formula $(L_j^{-\lambda} T) \underset{\lambda}{*} \varphi = T \underset{\lambda}{*} L_j^{\lambda} \varphi$ for $\varphi \in \mathcal{S}(E)$, *which is easy to prove.*

REMARK. *If $T \in \mathcal{O}'_{\lambda}(E)$*, then $\check{T}$, $\dfrac{\partial T}{\partial x_j}$ *and $P(x)T$ belong to $\mathcal{O}'_{\lambda}(E)$, where $P(x)$ is a polynomial. Furthermore, $\overline{T}$ and T^* $(T^*(x) = \overline{T(-x)})$ belong to $\mathcal{O}'_{-\lambda}(E)$ and we have: $\overline{S \underset{\lambda}{*} T} = \overline{T} \underset{\lambda}{*} \overline{S} = \overline{S} \underset{-\lambda}{*} \overline{T}$.*

PROPOSITION 7. *The twisted convolution (for λ) of tempered distributions is defined if all of them, except at most one of them denoted by S, belong to $\mathcal{O}'_{\pm\lambda}(E)$, these being on the right side of S belonging to $\mathcal{O}'_{\lambda}(E)$, the other ones belonging to $\mathcal{O}'_{-\lambda}(E)$. When it is defined, the twisted convolution of tempered distributions is associative.*

Proof. The prove the associativity, we need the following lemmas:

LEMMA 7.

1) *If S and T belong to $\mathcal{O}'_{\lambda}(E)$, then $S \underset{\lambda}{*} T$ belongs to $\mathcal{O}'_{\lambda}(E)$; and for every function φ of $\mathcal{S}(E)$, we have:*

$$S \underset{\lambda}{*} (T \underset{\lambda}{*} \varphi) = (S \underset{\lambda}{*} T) \underset{\lambda}{*} \varphi.$$

2) *Similarly, if S and T belong to $\mathcal{O}'_{-\lambda}(E)$, we have for every function φ of $\mathcal{S}(E)$:*

$$(\varphi \underset{\lambda}{*} S) \underset{\lambda}{*} T = \varphi \underset{\lambda}{*} (S \underset{\lambda}{*} T).$$

The proof of this lemma is obvious.

LEMMA 8. *If S belongs to $\mathcal{O}'_\lambda(E)$, T belongs to $\mathcal{O}'_{-\lambda}(E)$ and φ belongs to $\mathcal{S}(E)$, we have:*

$$(S \underset{\lambda}{*} \varphi) \underset{\lambda}{*} T = S \underset{\lambda}{*} (\varphi \underset{\lambda}{*} T).$$

Indeed, let us use an approximation of the identity in $\mathcal{S}(E)$ (see lemma 4), then $S \underset{\lambda}{*} \varphi_n$ and $\varphi \underset{\lambda}{*} T$ belong to $\mathcal{S}(E)$. By using lemma 6 for λ and $-\lambda$, we obtain:

$$S \underset{\lambda}{*} [\varphi_n \underset{\lambda}{*} (\varphi \underset{\lambda}{*} T)] = (S \underset{\lambda}{*} \varphi_n) \underset{\lambda}{*} (\varphi \underset{\lambda}{*} T) =$$
$$= [(S \underset{\lambda}{*} \varphi_n) \underset{\lambda}{*} \varphi] \underset{\lambda}{*} T = [S \underset{\lambda}{*} (\varphi_n \underset{\lambda}{*} \varphi)] \underset{\lambda}{*} T.$$

Lemma 8 is then obtained by taking the limit as n tends to infinity, with the help of lemma 4 and proposition 5,2).

Let us now return to the proof of proposition 7. Let us consider for example the case: $S_1 \in \mathcal{O}'_{-\lambda}(E)$, $S_2 \in \mathcal{O}'_\lambda(E)$ and $S \in \mathcal{S}'(E)$. Then by using theorem 1, we obtain:

$$\langle (S_1 \underset{\lambda}{*} S) \underset{\lambda}{*} S_2, \varphi \rangle = \langle S_1 \underset{\lambda}{*} S, \check{S}_2 \underset{\lambda}{*} \varphi \rangle = \langle S, (\check{S}_2 \underset{\lambda}{*} \varphi) \underset{\lambda}{*} \check{S}_1 \rangle$$

on one hand and:

$$\langle S_1 \underset{\lambda}{*} (S \underset{\lambda}{*} S_2), \varphi \rangle = \langle S \underset{\lambda}{*} S_2, \varphi \underset{\lambda}{*} \check{S}_1 \rangle = \langle S, \check{S}_2 \underset{\lambda}{*} (\varphi \underset{\lambda}{*} \check{S}_1 \rangle$$

on the other hand.

We then use lemma 8 to complete the proof. ∎

4. FOURIER TRANSFORM OF THE TWISTED CONVOLUTION PRODUCT. INTEGRAL FORMULA OF THE STAR PRODUCT

Let us set for every real number $\lambda \neq 0$ and for every $T \in \mathcal{S}'(E)$: $T^\lambda(x) = |\lambda|^{2l} \cdot T(\lambda x)$. Let us note that $\langle T^\lambda, \varphi \rangle = \langle T(x), \varphi\left(\frac{x}{\lambda}\right) \rangle$ if $\varphi \in \mathcal{S}(E)$. We have the lemma:

LEMMA 9. *For every tempered distribution T in E, every function φ of $\mathcal{S}(E)$ and every real number $\lambda \neq 0$, we have:*

$$T \underset{\lambda}{*} \hat{\varphi}(x) = \hat{T} \underset{1/\lambda}{*} \check{\varphi}(\lambda x).$$

Proof. Indeed, we have:

$$\hat{T} \underset{1/\lambda}{*} \varphi(x) = < T(y), e^{i\omega(x,y)} \cdot \hat{\varphi}\left(y - \frac{x}{\lambda}\right) >$$

and lemma 9 is obtained by replacing x by λx. ■

PROPOSITION 8. *Let λ be a real number different from zero, then T belongs to $\mathcal{O}'_\lambda(E)$ if and only if $\hat{T}$ and $T^{\pm 1/\lambda}$ belong to $\mathcal{O}'_{1/\lambda}(E)$; and if T belongs to $\mathcal{O}'_\lambda(E)$ or if S belongs to $\mathcal{O}'_{-\lambda}(E)$, we have:*

$$\mathcal{F}(S \underset{\lambda}{*} T) = S^{1/\lambda} \underset{1/\lambda}{*} \hat{T} = \hat{S} \underset{1/\lambda}{*} T^{-1/\lambda}.$$

Proof. Indeed, according to lemma 9, $T \in \mathcal{O}'_\lambda(E)$ if and only if $\hat{T} \in \mathcal{O}'_{1/\lambda}(E)$ and under this hypothesis we have:

$$\langle \mathcal{F}(S \underset{\lambda}{*} T), \varphi\rangle = \langle \check{S}, T \underset{\lambda}{*} \hat{\varphi}\rangle = \langle \check{S}(x), \hat{T} \underset{1/\lambda}{*} \check{\varphi}(\lambda x)\rangle = \langle S^{-1/\lambda}, \hat{T} \underset{1/\lambda}{*} \check{\varphi}\rangle$$

(The formula $(T^\lambda)^\vee = T^{-\lambda}$ has been used). The first equality in proposition 8 is obtained. Then, writing this first result in the form:

$$\mathcal{F}(T \underset{\lambda}{*} \varphi) = T^{1/\lambda} \underset{1/\lambda}{*} \hat{\varphi} \quad \text{for} \quad \varphi \in \mathcal{S}(E),$$

we see that T belongs to $\mathcal{O}'_\lambda(E)$ if and only if $T^{1/\lambda}$ and therefore also $T^{-1/\lambda}$ belong to $\mathcal{O}'_{1/\lambda}(E)$, and if $T \in \mathcal{O}'_\lambda(E)$ we have:

$$\langle \mathcal{F}(S \underset{\lambda}{*} T), \varphi\rangle = \langle \hat{S}, \mathcal{F}(T \underset{\lambda}{*} \hat{\varphi})\rangle = \langle \hat{S}, T^{1/\lambda} \underset{1/\lambda}{*} \varphi\rangle = \langle \hat{S} \underset{1/\lambda}{*} T^{-1/\lambda}, \varphi\rangle.$$

A similar computation is done if $T \notin \mathcal{O}'_\lambda(E)$, by using the hypothesis $S \in \mathcal{O}'_{-\lambda}(E)$. The proof of proposition 8 is then complete. ■

We deduce from proposition 8, that the star product $S \circ T$ is defined if $S \in \mathcal{O}'_{-2/\hbar}(E)$ or if $T \in \mathcal{O}'_{2/\hbar}(E)$, and we have the proposition:

PROPOSITION 9. *Integral formula of the star product.*

If S and T are two tempered distributions in E such that S belongs to $\mathcal{O}'_{-2/\hbar}(E)$ or T belongs to $\mathcal{O}'_{2/\hbar}(E)$, then $S \circ T$ is defined and satisfies the formulas:

$$\langle S \circ T, \varphi\rangle = \langle S \underset{2/\hbar}{*} \check{T}, (\hat{\varphi})^{-2/\hbar}\rangle$$

$$\langle S \circ T, \varphi\rangle = \left(\frac{2}{\hbar}\right)^{2l} < S(x), < T(y), \int \exp\left(2\, \frac{i}{\hbar}\, \omega(x - z, y - z)\right)\varphi(z)\, d\mu(z)>>$$

In the second formula, $S(x)$ and $T(y)$ can be exchanged (or must be exchanged) if S belongs to $\mathcal{O}'_{-2/\hbar}(E)$.

Proof. Indeed, according to proposition 8, we have:

$$\langle S \circ T, \varphi\rangle = \langle \mathscr{F}(\hat{S} \underset{\hbar/2}{*} \hat{T}), \varphi\rangle = \langle \mathscr{F}\check{S}, \hat{T} \underset{\hbar/2}{*} \hat{\varphi}\rangle =$$
$$= \langle S, \mathscr{F}(\hat{T} \underset{\hbar/2}{*} \hat{\varphi})\rangle = \langle S, T \underset{2/\hbar}{*} (\hat{\varphi})^{-2/\hbar}\rangle.$$

If $T \in \mathscr{O}'_{2/\hbar}(E)$, this formula becomes:

$$\langle S \circ T, \varphi\rangle = \left(\frac{2}{\hbar}\right)^{2l} < S(x), < T(y), \exp 2\, \frac{i}{\hbar}\, \omega(x,y) \int \exp\left(-2\, \frac{i}{\hbar}\, \omega(x-y,z)\right) \cdot \varphi(z)\, \mathrm{d}\mu(z) >>$$

Then, let us note that $\omega(x,y) - \omega(x-y,z) = \omega(x-z, y-z)$ and the proof is complete. ■

5. STRUCTURE OF THE DISTRIBUTIONS OF THE SPACE $\mathscr{O}'_\lambda$(E)

To study the structure of the distributions of $\mathscr{O}'_\lambda(E)$, we will use a lemma similar to the theorem XXII (chapter VI) of [12].

LEMMA 10. *Let λ be a real number and let B' be a bounded set of distributions of $\mathscr{D}'(E)$, then for every open subset Ω of E with compact closure, there is a positive integer m such that, for every function α of $\mathscr{D}^m(E)$ with compact support contained in the unit ball of E, the set of the distributions $T_j \underset{\lambda}{*} \alpha$, $T_j \in B'$, is equal on Ω to a uniformly bounded set of continuous functions.*

Proof. We give here a direct proof of this lemma. Indeed, Ω is contained in the ball $B(0,R)$ with center at zero and radius $R > 0$, and there is an index J such that the distributions T_j of B' are equal on $B(0, R+1)$ to $D_J f_j$, where the functions f_j are continuous with compact support in $B(0, R+2)$ and uniformly majorized by a positive number M (see [12], chapter III). Then if $m \geqslant |J|$, for every $\alpha \in \mathscr{D}^m(E)$ with compact support in $B(0,1)$ and for every $\varphi \in \mathscr{D}(E)$ with compact support in $B(0,R)$, we have:

$$\langle T_j \underset{\lambda}{*} \alpha, \varphi\rangle = (-1)^{|J|} \int f_j(x)\, \mathrm{d}\mu(x) \int D_{x^J}(e^{-i\lambda\omega(x,y)} \cdot \alpha(y-x)) \cdot \varphi(y)\, \mathrm{d}\mu(y).$$

We obtain for x in $B(0,R)$

$$T_j \underset{\lambda}{*} \alpha(x) = (-1)^{|J|} \int f_j(y) D_{y^J}(e^{i\lambda\omega(x,y)} \cdot \alpha(x-y))\, \mathrm{d}\mu(y)$$

it follows that $T_j \underset{\lambda}{*} \alpha(x)$ is continuous on $B(0,R)$ and that for x in $B(0,R)$:

$$|T_j \underset{\lambda}{*} \alpha(x)| \leqslant M\mu(B(0,R+2)) \sup_{\|x\| \leqslant R, \|y\| \leqslant R+2} |D_{y,J}(e^{i\lambda\omega(x,y)} \cdot \alpha(x-y))|. \qquad ■$$

THEOREM 2. *Let λ be a real number and let T be a distribution of $\mathscr{D}'(E)$, then the following three properties are equivalent:*

1) $T \in \mathcal{O}'_\lambda(E)$
2) $T \underset{\lambda}{*} \varphi \in \mathscr{S}(E)$, $\forall \varphi \in \mathscr{D}(E)$
3) *For every nonnegative integer r, T can be written in the form:*

$$T = \Sigma_J L_J^{-\lambda} \frac{f_J(x)}{(1+\|x\|^2)^r} \qquad \textit{(finite sum)},$$

where the functions f_J are continuous and bounded on E.

Proof. Indeed, in view of the definition of $\mathcal{O}'_\lambda(E)$, 1) implies 2). Furthermore, it follows immediatly from 3) that $T \in \mathscr{S}'(E)$ and that $T \underset{\lambda}{*} \varphi \in \mathscr{S}(E)$ for every $\varphi \in \mathscr{S}(E)$. So the assertion: 3) implies 1) is proved. Let us prove now that 2) implies 3). As in the proof of proposition 5, it follows from 2) that for every integer $r > 0$, the set of the distributions T_h, $h \in E$, with

$$T_h(x) = (1+\|h\|^2)^r \tau_{-h} [e^{i\lambda\omega(h,x)} \cdot T(x)],$$

is a bounded subset of $\mathscr{D}'(E)$.

By using lemma 10, we see that there is an integer $m > 0$ such that, for every function $\alpha \in \mathscr{D}^m(E)$ with compact support in $B(0,1)$, the set of the distributions $T_h \underset{\lambda}{*} \alpha$, $h \in E$ is equal on $B(0,1)$ to a set of continuous functions, uniformly majorized by a positive number M. Let us note that:

$$[\tau_{-h}(e^{i\lambda\omega(h,x)} \cdot T(x))] \underset{\lambda}{*} \alpha = \tau_{-h}[e^{i\lambda\omega(h,x)} \cdot T \underset{\lambda}{*} \alpha(x)]$$

then for every $h \in E$, the distribution $(1+\|h\|^2)^r \, e^{i\lambda\omega(h,x)} \cdot T \underset{\lambda}{*} \alpha(x)$ is equal on $B(h,1)$ to a continuous function majorized by M. It follows that $T \underset{\lambda}{*} \alpha$ is a continuous function on E which satisfies the formula:

$$\sup_{x \in E} (1+\|x\|^2)^r \, |T \underset{\lambda}{*} \alpha(x)| \leqslant M.$$

Furthermore, we deduce from the formula (VI,6,22, p. 191) of [12] that there is a sufficiently large positive integer k and that there are $\alpha \in \mathscr{D}^m(E)$ and $\xi \in \mathscr{D}(E)$ with compact support in $B(0,1)$ such that $\Delta^k \alpha - \xi = \delta$. Let us use now the formula (6-b) of lemma 2, we obtain: $\Delta^k = \sum_J L_J^{-\lambda}(P_J(x)\alpha(x))$, where the functions $P_J(x)$ are polynomials, then the functions $P_J(x)\alpha(x)$ belong to $\mathscr{D}^m(E)$ and we have:

$$T = T \underset{\lambda}{*} \delta = \sum_J T \underset{\lambda}{*} L_J^{-\lambda}(P_J(x)\,\alpha(x)) - T \underset{\lambda}{*} \xi$$

Let us use now proposition 6,2), which still holds for the convolution in $\mathscr{D}'(E)$. Theorem 2 is then proved. ■

REMARK 1. *The space $\xi'(E)$ of distributions with compact support in E is contained in $\mathscr{O}'_\lambda(E)$ for every real number λ. The space $\mathscr{F}(\xi'(E))$ and in particular the space of polynomials on E are contained in $\mathscr{O}'_\lambda(E)$ for every real number $\lambda \neq 0$.*

To see that, it suffices to consider the structure of the distributions of $\xi'(E)$ (see [12], chapter III) and to apply theorem 2 and proposition 8.

REMARK 2. *The space of infinitely differentiable functions in E such that there is a positive integer m (depending from α but not from the index J), sucht that for every index J, we have: $|D_J\alpha(x)| \leqslant A_J(1 + \|x\|^2)^m$, is contained in $\mathscr{O}'_\lambda(E)$ for every real number $\lambda \neq 0$, as well as the Fourier transforms of these functions.*

Proof. Indeed, it suffices to use the following formula which holds if we use a canonical basis:

$$\mathscr{F}\alpha(x) = (1 - \Delta_x)^k \frac{1}{(1 + \|x\|^2)^r} \mathscr{F}\left[(1 - \Delta_y)^r \frac{\alpha(y)}{(1 + \|y\|^2)^k}\right]$$

and to note that there is a sufficiently large positive integer k such that $\mathscr{F}\alpha(x)$ can be written in the form:

$$\mathscr{F}\alpha(x) = (1 - \Delta_x)^k \frac{f_r(x)}{(1 + \|x\|^2)^r}$$

for every positive integer r, where the function f_r is continuous and bounded, and it suffices then to apply the formula (6-b) of lemma 2 so as to obtain:

$$\mathscr{F}\alpha(x) = \sum_J L_J^{-\lambda} \frac{P_J(x) f_r(x)}{(1 + \|x\|^2)^r} \qquad \text{(finite sum)},$$

where the functions $P_J(x)$ are polynomials of degree not greater than $2k$, k being not depending from r, and it suffices then to apply theorem 2. ■

REMARK 3. *$\mathscr{O}_M(E)$ and $\mathscr{O}'_C(E)$ are not contained in $\mathscr{O}'_\lambda(E)$ (see §7).*

PROPOSITION 10. *Let λ be a real number, then a distribution T of $\mathscr{D}'(E)$ belongs to $\mathscr{S}'(E)$ if and only if $T \underset{\lambda}{*} \varphi$ belongs to $\mathscr{O}_M(E)$ for every function φ of $\mathscr{D}(E)$.*

The proof of this proposition is similar to that of the theorem VI, 2° (chapter VII, p. 241) of [12].

PROPOSITION 11. *Let λ be a real number and let T be a distribution of $\mathscr{D}'(E)$, then the two following properties are equivalent:*

1) $T \underset{\lambda}{*} \varphi \in L^2(E), \forall \varphi \in \mathscr{D}(E)$
2) *T can be written in the form:*

$$T = \sum_J L_J^{-\lambda} f_J \qquad (\textit{finite sum}),$$

where the functions f_J belong to $L^2(E)$.

Under these hypotheses, T belongs to $\mathscr{S}'(E)$ and $T \underset{\lambda}{*} \varphi$ belongs to $L^2(E)$ for every function φ of $\mathscr{S}(E)$.

Proof. We prove this proposition with similar arguments to those used in the proof of the theorem XXV (chapter VI) of [12]. Indeed, hypothesis 2) clearly implies that $T \in \mathscr{S}'(E)$. Let us endow $\mathscr{S}(E)$ with the thopology induced by the family of seminorms $\rho_J(\varphi) = \| L_J^{\lambda} \varphi \|_2$ for every index J, where $\| \;\; \|_2$ denotes the norm of the elements of $L^2(E, \mu)$. Then T is continuous for this topology. Let B be the set of the functions $\alpha \in \mathscr{D}(E)$ such that $\| \alpha \|_2 \leqslant 1$, B is dense in the unit ball of $L^2(E)$. Then, for every function φ of $\mathscr{S}(E)$, we have:

$$\sup_{\alpha \in B} | \langle T \underset{\lambda}{*} \varphi, \alpha \rangle | = \sup_{\alpha \in B} | \langle T, \check{\varphi} \underset{\lambda}{*} \alpha \rangle | < +\infty \tag{11}$$

since the set of the functions $\check{\varphi} \underset{\lambda}{*} \alpha$, $\alpha \in B$ is a bounded subset of $\mathscr{S}(E)$ for the topology previously induced in $\mathscr{S}(E)$. Indeed, from the formula (9-b) of proposition 3, and from the well known properties of the twisted convolution of functions (see [4]), we deduce:

$$\| L_J^{\lambda} (\check{\varphi} \underset{\lambda}{*} \alpha) \|_2 \leqslant \| L_J^{\lambda} \check{\varphi} \|_1 \| \alpha \|_2 \leqslant \| L_J^{\lambda} \check{\varphi} \|_1 .$$

Then, formula (11) implies that $T \underset{\lambda}{*} \varphi \in L^2(E)$.

Conversely, let us suppose that $T \underset{\lambda}{*} \varphi \in L^2(E)$, $\forall \varphi \in \mathscr{D}(E)$, then we deduce from the formula:

$$\alpha \underset{\lambda}{*} T \underset{\lambda}{*} \check{\varphi}(0) = \langle \alpha \underset{\lambda}{*} T, \varphi \rangle = \langle T \underset{\lambda}{*} \check{\varphi}, \check{\alpha} \rangle$$

that for every function φ of $\mathscr{D}(E)$, we have: $\sup_{\alpha \in B} | \langle \alpha \underset{\lambda}{*} T, \varphi \rangle | < +\infty$. The set of the distributions $\alpha \underset{\lambda}{*} T$, $\alpha \in B$ is then a bounded subset of $\mathscr{D}'(E)$. Let us use lemma 10, and let $\varphi \in \mathscr{D}^m(E)$ with a compact support in $B(0, 1)$ (m chosen as in lemma 10). Then, on a neighborhood of zero with compact closure, the distributions $\alpha \underset{\lambda}{*} T \underset{\lambda}{*} \varphi$, $\alpha \in B$ are equal to continuous functions, uniformly bounded on this neighborhood. By considering the values of these functions at zero, we obtain the formula:

$$\sup_{\alpha \in B} |\langle T \underset{\lambda}{*} \varphi, \check{\alpha} \rangle| < +\infty .$$

It follows that $T \underset{\lambda}{*} \varphi \in L^2(E)$.

Then, as in the proof of theorem 2, it suffices to use the formula (VI.6.22) of [12] to obtain proposition 11. ∎

6. WEYL TRANSFORMS OF THE DISTRIBUTIONS OF $\mathscr{O}'_{\hbar/2}(E)$. BOUNDED OPERATORS OF $\mathbb{H}$

The purpose of this section is to characterize the symbols of the operators of $\mathscr{L}(\mathbb{H}_\infty, \mathbb{H}_\infty)$ as well as the symbols of the bounded operators of $\mathbb{H}$, where $\mathbb{H}$ is the Hilbert space of an irreducible unitary representation of the Weyl group associated with $\hbar \neq 0$. For this purpose, we can take the representation of the Weyl group on $L^2(\mathbb{R}^l)$, defined by formula (4). Let us keep in mind that $\mathbb{H}_\infty = \mathscr{S}(\mathbb{R}^l)$ in this case.

Let us first note that for all $u, v \in \mathbb{H}_\infty$, we have:

$$(\mathrm{op}(T)\, u \,|\, \pi(-\hbar x, 0)\, v) = \hat{T} \underset{\hbar/2}{*} \check{\chi}_{u,v}(x). \tag{12}$$

We will use the following lemmas.

LEMMA 11. *Let L be a tempered distribution of $\mathscr{S}'(\mathbb{R}^l)$ and u be a function of $\mathscr{S}(\mathbb{R}^l)$, then the function $\beta(p) = \langle L(q), e^{-ipq} \cdot u(q) \rangle$, $p, q \in \mathbb{R}^l$ and the distribution $\mathscr{F}(uL)$ (where $\mathscr{F}$ denotes the usual Fourier transform on $\mathbb{R}^l$) belong to the space $\mathscr{O}_M(\mathbb{R}^l)$ and are equal.*

Proof. Indeed, for every $\varphi \in \mathscr{S}(\mathbb{R}^l)$, we have:

$$\langle \mathscr{F}(uL), \varphi \rangle = \left\langle L(q), u(q) \int e^{-ipq} \cdot \varphi(p)\, \mathrm{d}m(p) \right\rangle =$$

$$= \int \varphi(p) \, \langle L(q), u(q)\, e^{-ipq} \rangle \, \mathrm{d}m(p)$$

where $dm(p) = (2\pi)^{-l/2} \cdot dp$. The inversion under the integral sign is justified by using the structure of tempered distributions. ■

LEMMA 12. *Let L be a tempered distribution of $\mathcal{S}'(\mathbb{R}^l)$ such that, for every function u of $\mathcal{S}(\mathbb{R}^l)$, the function $f(x) = L(\pi(\hbar x, 0)u)$ belongs to $\mathcal{S}(E)$, then L belongs to $\mathcal{S}(\mathbb{R}^l)$.*

Proof. Indeed, if $x = q + p$, we have:

$$f(q, p) = \left\langle L(q'), \exp\left(-i\left(\frac{\hbar}{2}pq + pq'\right)\right) \cdot u(q' + \hbar q)\right\rangle \tag{13}$$

where $f(q, p) \in \mathcal{S}(E)$. It is clear that $f(o, p) = \langle L(q'), e^{-ipq'} \cdot u(q')\rangle$ belongs to $\mathcal{S}(\mathbb{R}^l)$, and from lemma 11, we deduce:

$$\mathcal{F}(uL) = \hat{u} * \hat{L} \in \mathcal{S}(\mathbb{R}^l), \quad \forall u \in \mathcal{S}(\mathbb{R}^l)$$

($\mathcal{F}$ denotes here the usual Fourier transform and * denotes the usual convolution product). Then $\hat{L} \in \mathcal{O}'_C(\mathbb{R}^l)$, so $L \in \mathcal{O}_M(\mathbb{R}^l)$. We then deduce from formula (13) and from the Fourier inversion formula with respect to q', that:

$$h(q, q') = L(q')u(q' + \hbar q) = \int \exp\left(i\left(q'p + \frac{\hbar}{2}pq\right)\right) \cdot f(q, p)\, dm(p)$$

belongs to $\mathcal{S}(\mathbb{R}^{2l})$. Then $u(o)\, L(q') = h\left(-\frac{q'}{\hbar}, q'\right) \in \mathcal{S}(\mathbb{R}^l)$. We complete the proof by choosing $u \in \mathcal{S}(\mathbb{R}^l)$ such that $u(o) \neq 0$. ■

THEOREM 3. *Let T be a tempered distribution of $\mathcal{S}'(E)$, then* op(T) *belongs to $\mathcal{L}(\mathbb{H}_\infty, \mathbb{H}_\infty)$ if and only if $\hat{T}$ belongs to $\mathcal{O}'_{\hbar/2}(E)$. And in this case, for every admissible symbol S, $S \circ T$ is an admissible sybol and we have:*

$$\mathrm{op}\,(S \circ T) = \mathrm{op}\,(S)\,\mathrm{op}\,(T).$$

Proof. Indeed, if $\hat{T} \in \mathcal{O}'_{\hbar/2}(E)$, we deduce from formula (12) and from lemma 12 that $\mathrm{op}\,(T)u \in \mathbb{H}_\infty$, $\forall u \in \mathbb{H}_\infty$. It follows from the closed graph theorem that $\mathrm{op}\,(T)$ is continuous of $\mathbb{H}_\infty$ into itself.

Conservely, if $\mathrm{op}\,(T) \in \mathcal{L}(\mathbb{H}_\infty, \mathbb{H}_\infty)$, formula (12) becomes:

$$\check{\chi}_{\mathrm{op}(T)u,v} = \hat{T} \underset{\hbar/2}{*} \check{\chi}_{u,v}. \tag{14}$$

We then have $\hat{T} \underset{\hbar/2}{*} \varphi \in \mathcal{S}(E)$ for every function φ belonging to the dense linear subspace $\tilde{\mathcal{S}}$ of $\mathcal{S}(E)$ of the functions written in the form:

$$\Sigma\, \check{\chi}_{u_j, v_j} = [\,\mathscr{C}(\Sigma\, u_j \otimes \overline{v}_j)]^{\vee} \qquad \text{(finite sum)},$$

where $u_j,\ v_j \in \mathscr{S}(\mathbb{R}^l)$ and where $\mathscr{C}$ is defined by formula (5). Furthermore, we deduce from the continuity of op (T) that the mapping

$$\Sigma\, u_j \otimes \overline{v}_j \to \Sigma\ \mathrm{op}\,(T)\, u_j \otimes \overline{v}_j \qquad \text{(finite sum)}$$

is a continuous linear map of the space $\mathscr{S}(\mathbb{R}^l) \otimes \mathscr{S}(\mathbb{R}^l)$ into itself, if this space is endowed with the π topology (see [13], chapter 43) which is equal to the topology induced by that of $\mathscr{S}(\mathbb{R}^{2l})$ (see [13], proof of theorem 51-6, p. 531). $\mathscr{C}$ being an isomorphism of $\mathscr{S}(E)$ onto itself, we then deduce from the following sequence of continuous linear maps:

$$\check{\varphi} = \Sigma\, \chi_{u_j, v_j} \to \Sigma\, u_j \otimes \overline{v}_j \to \Sigma\ \mathrm{op}\,(T)\, u_j \otimes \overline{v}_j \to \Sigma\, \chi_{\mathrm{op}\,(T) u_j, v_j} = (\hat{T} \underset{\hbar/2}{*} \varphi)^{\vee}$$

that $\varphi \to \hat{T} \underset{\hbar/2}{*} \varphi$ is a continuous linear map of $\tilde{\mathscr{S}}$ (equipped with the topology induced by that of $\mathscr{S}(E)$) into $\mathscr{S}(E)$. This map can be extended to a continuous linear map of $\mathscr{S}(E)$ into itself. We then obtain $\hat{T} \underset{\hbar/2}{*} \varphi \in \mathscr{S}(E)$, $\forall\, \varphi \in \mathscr{S}(E)$ and the proof is complete. ∎

By using proposition 8, we clearly deduce from theorem 3, the following corollary:

COROLLARY 2. *Let T be a tempered distribution of $\mathscr{S}'(E)$, then the operator* op (T) *belongs to $\mathscr{L}(\mathbb{H}_\infty, \mathbb{H}_\infty)$ if and only if T belongs to the space $\mathcal{O}'_{2/\hbar}(E)$.*

REMARK. *We deduce from* [13], *proposition* 50.4, *that* op (T) *belongs to $\mathscr{L}(\mathscr{S}(\mathbb{R}^l), \mathscr{S}(\mathbb{R}^l))$ if and only if its kernel belongs to $\mathscr{S}'(\mathbb{R}^l) \hat{\otimes} \mathscr{S}(\mathbb{R}^l)$. So, we obtain the characterization of the kernels of the elements of $\mathcal{O}'_{\hbar/2}(E)$, previously established in* [11], *theorem 1.*

THEOREM 4. *A tempered distribution T of $\mathscr{S}'(E)$ is an admissible symbol for the Weyl transformation if and only if $\hat{T} \underset{\hbar/2}{*} \varphi$ belongs to $L^2(E)$ for every function φ of $\mathscr{S}(E)$.*

REMARK. *If T is an admissible symbol, the linear map $\varphi \to \hat{T} \underset{\hbar/2}{*} \varphi$ is then continuous of $\mathscr{S}(E)$ into $L^2(E)$.*

Proof. Indeed, let us suppose that T is an admissible symbol, then op $(T)\, u \in \mathbb{H}$, $\forall\, u \in \mathbb{H}_\infty$. Formula (14) (which still holds in this case) and lemma 1 show that $\hat{T} \underset{\hbar/2}{*} \varphi \in L^2(E)$, $\forall\, \varphi \in \tilde{\mathscr{S}}$. The operator op (T) being continuous of $\mathscr{S}(\mathbb{R}^l)$ into $L^2(\mathbb{R}^l)$ and the map:

$$\Sigma\, u_j \otimes \bar{v}_j \to \Sigma\ \mathrm{op}\,(T)\, u_j \otimes \bar{v}_j$$

being therefore continuous of $\mathscr{S}(\mathbb{R}^l) \otimes \mathscr{S}(\mathbb{R}^l)$ (endowed with the π topology which is identical with the topology induced by that of $\mathscr{S}(\mathbb{R}^{2l})$) into $L^2(\mathbb{R}^l) \otimes \otimes L^2(\mathbb{R}^l)$ (equipped with the π topology), we deduce from Lemma 1 that the map $\varphi \to \hat{T} \underset{\hbar/2}{*} \varphi$ is continuous of $\tilde{\mathscr{S}}$ (equipped with the topology induced by that of $\mathscr{S}(E)$) into $L^2(E)$ and therefore can be extended to a continuous linear map of $\mathscr{S}(E)$ into $L^2(E)$. Then $\hat{T} \underset{\hbar/2}{*} \varphi \in L^2(E)$, $\forall \varphi \in \mathscr{S}(E)$.

Conversely, let us suppose that $\hat{T} \underset{\hbar/2}{*} \varphi \in L^2(E)$, $\forall \varphi \in \mathscr{S}(E)$. Let $u \neq 0$ be a function of $\mathscr{S}(\mathbb{R}^l)$. Let us consider the bounded operator A of $L^2(\mathbb{R}^l)$ defined by: $Av = (v \mid u) \dfrac{u}{\| u \|^2}$, $v \in L^2(\mathbb{R}^l)$. The kernel of this operator belonging to $\mathscr{S}(E)$, its symbol f also belongs to $\mathscr{S}(E)$ and $\hat{T} \underset{\hbar/2}{*} \hat{f} \in L^2(E)$. Then, $\mathrm{op}\,(T \circ f) = \mathrm{op}\,(T)\, A$ is a bounded operator of $L^2(\mathbb{R}^l)$ and $\mathrm{op}\,(T)u = \mathrm{op}\,(T \circ f)u \in L^2(\mathbb{R}^l)$. We obtained $\mathrm{op}\,(T)(\mathbb{H}_\infty) \subset \mathbb{H}$ and T is then an admissible symbol. ■

By using lemma 9 and proposition 11, we obtain the following corollary:

COROLLARY 3. *A tempered distribution T of $\mathscr{S}'(E)$ is an admissible symbol for the Weyl transformation if and only if one of the two following equivalent properties is satisfied:*

1) $T \underset{2/\hbar}{*} \varphi \in L^2(E)$, $\forall \varphi \in \mathscr{D}(E)$
2) *T can be written in the form:*

$$T = \sum_J L_J^{-2/\hbar} f_J \qquad \textit{(finite sum)},$$

where the functions f_J belong to $L^2(E)$.

REMARK. *Let us note that if $\lambda \neq 0$, $\mathscr{F} \circ L_j^{\lambda} = \lambda\, L_j^{1/\lambda} \circ \mathscr{F}$.*

Our purpose is now to characterize the symbols of the bounded operators of $\mathbb{H}$.

DEFINITION 3. *A tempered distribution T of $\mathscr{S}'(E)$ is called a bounded λ-convolution operator of $L^2(E)$ if and only if $T \underset{\lambda}{*} \varphi$ belongs to $L^2(E)$ for every function φ of $\mathscr{S}(E)$ and if there is a nonnegative real number M such that $\| T \underset{\lambda}{*} \varphi \|_2 \leqslant \leqslant M \| \varphi \|_2$ for every function φ of $\mathscr{S}(E)$.*

We denote by $\| T \|_c$ the lower bound of the numbers M satisfying the preceding condition.

THEOREM 5. *The bounded operators of* $\mathbb{H}$ *are the Weyl transforms of the tempered distributions* T *of* $\mathcal{S}'(E)$ *whose Fourier transforms are the bounded* $\hbar/2$*-convolution operators of* $L^2(E)$, *and we have* $\| \operatorname{op}(T) \| = \| \hat{T} \|_c$.

This result was announced in [8]. Let us note that the number λ in the definition of the twisted convolution product in [8], is replaced by $-\lambda$ in this paper.

Proof. Let us prove theorem 5. Every bounded operator A of $\mathbb{H}$ has a symbol $T \in \mathcal{S}'(E)$. It is known that the Weyl transforms of the elements of $L^2(E)$ are the Hilbert-Schmidt operators of $\mathbb{H}$ and that:

$$\| \operatorname{op}(f) \|_{HS} = | \hbar |^{-1/2} \| f \|_2$$

where $\| \;\; \|_{HS}$ denotes the Hilbert-Schmidt norm of these operators. If $f \in \mathcal{S}(E)$, $\operatorname{op}(f)$ and $\operatorname{op}(T \circ f) = A \operatorname{op}(f)$ are Hilbert-Schmidt operators and we have:

$$\| \operatorname{op}(T \circ f) \|_{HS} = \| A \operatorname{op}(f) \|_{HS} \leqslant \| A \| \| \operatorname{op}(f) \|_{HS}$$

where $\| A \|$ is the norm of the bounded operators A. It follows that $T \circ f$ as well as $\hat{T} \underset{\hbar/2}{*} \hat{f}$ belong to $L^2(E)$ and that $\| \hat{T} \underset{\hbar/2}{*} \hat{f} \|_2 \leqslant \| A \| \| \hat{f} \|_2$. Then, $\hat{T}$ is a bounded $\hbar/2$-convolution operator of $L^2(E)$ which satisfies $\| \hat{T} \|_c \leqslant \| A \|$.

Conversely, if $\hat{T}$ is a bounded $\hbar/2$-convolution operator of $L^2(E)$, then according to theorem 4, T is an admissible symbol and we deduce from formula (14) and from formula (1), lemma 1, that:

$$\| \operatorname{op}(T) u \| \cdot \| v \| \leqslant \| \hat{T} \|_c \cdot \| u \| \cdot \| v \|, \qquad \forall u, v \in \mathbb{H}_\infty .$$

Then, $\operatorname{op}(T)$ is bounded and satisfies $\| \operatorname{op}(T) \| \leqslant \| \hat{T} \|_c$. ∎

REMARK. *If* $\hat{T}$ *is an isometric* $\hbar/2$*-convolution operator of* $L^2(E)$, *then* $\operatorname{op}(T)$ *is an isometric operator of* $\mathbb{H}$.

We can define in every case the twisted convolution of two bounded $\hbar/2$-convolution operators $\hat{S}$ and $\hat{T}$ of $L^2(E)$ as the Fourier transform of the symbol of the bounded operator $\operatorname{op}(S) \operatorname{op}(T)$. So, we have the following corollary:

COROLLARY 4. *The space of the bounded* $\hbar/2$*-convolution operators of* $L^2(E)$, *equipped with the twisted convolution product associated with* $\hbar/2$ *and with the norm* $\| \;\; \|_c$, *is a Banach algebra isometric to the algebra of the bounded operators of* $\mathbb{H}$.

By using lemma 9, we deduce from theorem 5, the corollary:

COROLLARY 5. *The bounded operators of* $\mathbb{H}$ *are the Weyl transforms of the bounded* $2/\hbar$*-convolution operators of* $L^2(E)$ *and we have:* $\|\operatorname{op}(T)\| = (2/|\hbar|)^l \|T\|_c$.

REMARK. *It follows from the formulas* $\operatorname{op}(\overline{T}) = \operatorname{op}(T^*)$ *and* $\overline{T \underset{\lambda}{*} \varphi} = \overline{\varphi} \underset{\lambda}{*} \overline{T}$, *and from theorem 5, that every left bounded* λ*-convolution operator of* $L^2(E)$ *is also a right bounded* λ*-convolution operator of* $L^2(E)$.

If φ belongs to $\mathscr{S}(E)$, $\operatorname{op}(\varphi)$ is a trace class operator (see [5], theorem 7), Indeed, if $\varphi \in \mathscr{S}(E)$, the kernel of the operator $\operatorname{op}(\varphi)$ belongs to $\mathscr{S}(\mathbb{R}^{2l})$ and it is easy to see that $\operatorname{op}(\varphi)$ is a continuous linear map of $L^2(\mathbb{R}^l)$ into $\mathscr{S}(\mathbb{R}^l)$. The natural injection of $\mathscr{S}(\mathbb{R}^l)$ into $L^2(\mathbb{R}^l)$ being nuclear, it follows that $\operatorname{op}(\varphi)$ is a nuclear operator of $L^2(\mathbb{R}^l)$ into itself and therefore is a trace class operator (see [13], chapters 48 and 50). We have the proposition:

PROPOSITION 12. *Wigner transforms of the bounded operators.*

If A *is a bounded operator of* $\mathbb{H}$, *its symbol* T *is given by* $\langle T, \varphi\rangle = |\hbar|^l$ *trace* $(A \operatorname{op}(\varphi))$ *for every function* φ *of* $\mathscr{S}(E)$.

Proof. Indeed, for every function φ of $\mathscr{S}(E)$, $A \operatorname{op}(\varphi) = \operatorname{op}(T \circ \varphi)$ is a trace class operator. Then, the Fourier transform $\hat{T} \underset{\hbar/2}{*} \hat{\varphi}$ of its symbol belongs to $L^2(E)$ and is continuous (in fact, it can be written in the form: $\hat{T} \underset{\hbar/2}{*} \hat{\varphi} = f \underset{\hbar/2}{*} g$, $f, g\ L^2(E)$). (see [5], section 2) and we have $\operatorname{trace}(\operatorname{op}(T \circ \varphi)) = |\hbar|^{-l}\, \hat{T} \underset{\hbar/2}{*} \hat{\varphi}(0)$. It suffices therefore to apply proposition 3, formula 9-a. ■

7. APPLICATION TO STAR EXPONENTIALS

The purpose of this section is to apply the preceding results, so as to show that the Weyl transforms of the star exponentials obtained in [9], theorem 1, form one parameter groups of unitary operators. We will use the notations of [9].

Let $T(x) = e^{i(Ax|x)}$, where A is a real symmetric matrix with respect to a basis of E. This matrix is considered as a linear map of E into its dual E'. Let us denote by ν the linear map of E into E' defined by $\nu(x) = -i_x \omega$, $(\langle i_x \omega, y\rangle = \omega(x, y))$. For every function φ of $\mathscr{S}(E)$, we have:

$$T \underset{2/\hbar}{*} \varphi(x) = e^{i(Ax|x)}\, \mathscr{F}(e^{i(Ay|y)} \cdot \varphi(y))\left(2\left(A - \frac{\nu}{\hbar}\right)(x)\right)$$

where $\mathscr{F}\varphi$ denotes the usual Fourier transform of φ, considered as a function in E'. Then, by using corollaries 2, 3 and 5, as well as the remark following theorem

5, we obtain the following proposition, where $\det A$ denotes the determinant of the linear map A.

PROPOSITION 13. *Let* $T(x) = e^{i(Ax|x)}$, *where* A *is a real symmetric matrix. If* $\det\left(A - \frac{\nu}{\hbar}\right)$ *is different from zero, then* $\operatorname{op}(T)$ *is a bounded operator of* $\mathbb{H}$ *such that* $\operatorname{op}(T)(\mathbb{H}_\infty) \subset \mathbb{H}_\infty$, *which satisfies the formula:*

$$\|\operatorname{op}(T)u\| = |\hbar|^{-l}\left|\det\left(A - \frac{\nu}{\hbar}\right)\right|^{-1/2}\|u\|$$

for every vector u *of* $\mathbb{H}$.

If $\det\left(A - \frac{\nu}{\hbar}\right)$ *is equal to zero, then* T *is not an admissible symbol.*

Let us note that with respect to a canonical basis of E (that is a basis of the form $(e_i, e_{\bar{i}})$ such that $\omega(e_i, e_{\bar{i}}) = 1$, the other components being equal to zero) and with respect to the dual basis of E', the matrix of the linear map ν is equal to the matrix denoted by Λ in [9]. Let us note that trace $(\Lambda A) = 0$ and that: $|\det \cosh(t\Lambda A)| \cdot |\det(\tanh(t\Lambda A) - I)| = 1$; we then deduce from proposition 13 that the star exponentials obtained in [9], theorem 1, are bounded operators of $\mathbb{H}$, which are isometric (it will follow from proposition 14, that they are unitary), and such that the images of $\mathbb{H}_\infty$ under these operators are contained in $\mathbb{H}_\infty$. In fact, these star exponentials belong to $\mathcal{O}'_{2/\hbar}(E)$.

Let us note that the right hand side of the formula giving the value of $\exp^* tX$, in the theorem 1 of [9], is defined for all real numbers t if the eigenvalues of ΛA are not purely imaginary. It remains to prove that:

$$\exp^* sX \circ \exp^* tX = \exp^*(s+t)X.$$

The proof of this result requires the following lemmas:

LEMMA 13. *Let* f *and* g *be two functions of* $\mathcal{O}_M(E)$ *such that* $\mathcal{F}\left(f(x)g(y)\exp 2\frac{i}{\hbar}\omega(x,y)\right)$ *is continuous and slowly increasing at infinity, where* $\mathcal{F}$ *is the usual Fourier transformation on* $E \times E$. *Let us suppose that* f *belongs to* $\mathcal{O}'_{-2/\hbar}(E)$ *or* g *belongs to* $\mathcal{O}'_{2/\hbar}(E)$, *then* $f \circ g$ *is defined and satisfies the formula:*

$$f \circ g(x) = (2/\hbar)^{2l}\,\mathcal{F}\left(f(x)g(y)\exp 2\frac{i}{\hbar}\omega(x,y)\right)\left(\frac{2}{\hbar}\nu(x), -\frac{2}{\hbar}\nu(x)\right).$$

Proof. This lemma is easily deduced from the integral formula of the star product (proposition 9), by introducing the factor $\exp(-\|x\|^2/n^2)$ so as to change the order of integration, using the Fubini's theorem, and then by taking the limit as n tends to infinity, with the help of the Lebesgue's dominated convergence theorem. ∎

LEMMA 14. *If $\mathscr{A}$ is an invertible real symmetric matrix with n rows and n columns and if $f(x) = e^{i(\mathscr{A}x|x)}$, then:*

$$\mathscr{F}f(x) = \frac{\exp\left(i(\alpha-\beta)\frac{\pi}{4}\right)}{2^{n/2}|\det \mathscr{A}|^{1/2}} \exp\left(-\frac{i}{4}(\mathscr{A}^{-1}x|x)\right) \tag{15}$$

where α is the number of positive eigenvalues and β the number of negative eigenvalues of the matrix $\mathscr{A}$, and where $\mathscr{F}$ denotes the usual Fourier transformation.

This lemma is obtained by a straightforward computation.

It is now easy to obtain the following proposition.

PROPOSITION 14. *Let $X = (Ax|x)$, where A is a real symmetric matrix, then:*

$$\exp^* sX \circ \exp^* tX = \exp^* (s+t)X$$

(s and t are supposed sufficiently small if one of the eigenvalues of ΛA is purely imaginary).

Proof. Indeed, let us set $f_s(x) = \exp i(A_s x|x)$, where $A_s = \frac{1}{\hbar}\Lambda \tanh(s\Lambda A)$. We have:

$$f_s(x) f_t(y) \exp\left(2\frac{i}{\hbar}\omega(x,y)\right) = \exp i(\mathscr{A}_{s,t}(x+y)|x+y)$$

with:

$$\mathscr{A}_{s,t} = \begin{pmatrix} A_s & \frac{\Lambda}{\hbar} \\ -\frac{\Lambda}{\hbar} & A_t \end{pmatrix} = \frac{1}{\hbar}\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}\begin{pmatrix} \tanh(s\Lambda A) & I \\ -I & \tanh(t\Lambda A) \end{pmatrix}.$$

Let us use the well known lemma:

LEMMA 15. *If A, B, C and D are four $n \times n$ matrices such that C and D commute, then:*

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det (AD - BC).$$

Then we obtain: $\det \mathscr{A}_{s,t} = \hbar^{-4l} \det D_{s,t}$, where

$$D_{s,t} = I + \tanh (s \Lambda A) \tanh (t\Lambda A)$$

$$\mathscr{A}_{s,t}^{-1} = -\hbar \begin{pmatrix} \tanh (t \Lambda A) D_{s,t}^{-1} \Lambda & -D_{s,t}^{-1} \Lambda \\ D_{s,t}^{-1} \Lambda & \tanh (s \Lambda A) D_{s,t}^{-1} \cdot \Lambda \end{pmatrix}.$$

Proposition 14 is then deduced from lemmas 13 and 14, from the formulas:

$$\cosh (s + t) \Lambda A = \cosh (s \Lambda A) \cdot \cosh (t \Lambda A) D_{s,t}$$

$$\tanh (s + t) \Lambda A = (\tanh (s \Lambda A) + \tanh (t \Lambda A)) D_{s,t}^{-1}$$

and from the fact that $\alpha = \beta$ in formula (15) applied to $\mathscr{A}_{s,t}$. Indeed, this last property is easily verified for $s = t = 0$ (it suffices to apply lemma 15), for the other values of s and t, it suffices to apply the proposition 19 (chapter XII) of [14]. ∎

Finally, we prove a proposition which is required to justify the formal computations in [9]. Let $\nu(x) = -i_x \omega$ be the isomorphism of E onto E' previously introduced, this isomorphism is extended in a natural way to an isomorphism of the tensor algebra over E onto the tensor algebra over E'. Let $\Lambda = \nu^{-1}(\omega)$ (see [1]) and let us respectively denote by Λ_{jk} and by ω_{jk} the components of the 2-tensor Λ and of the 2-form ω with respect to a basis of E and to the dual basis of E'. We have $\sum_k \Lambda_{jk} \omega_{kl} = -\delta_{jl}$ and

$$\sum_{k,l} \Lambda_{jk} \omega_{kl} \Lambda_{jm} = -\Lambda_{jm} \tag{16}$$

Let us note that, with respect to a canonical basis of E (and to the dual basis of E'), the matrices (ω_{jk}) and (Λ_{jk}) are both equal to the matrix denoted by Λ in [9]. Let us consider the bidifferential operators (see [1] and [9]) defined for $f, g \in \mathscr{C}^\infty(E)$ by:

$$P_n(f, g) = \sum_{j_r, k_s = 1}^{2l} \Lambda_{j_1 k_1} \ldots \Lambda_{j_n k_n} \cdot D_{j_1 \ldots j_n} f \cdot D_{k_1 \ldots k_n} g.$$

We have the proposition:

PROPOSITION 15. *Let f and g be two tempered distributions of $\mathscr{S}'(E)$ such that one of them at least is a polynomial on E, then $f \circ g$ is defined and satisfies the formula:*

$$f \circ g = fg + \sum_{n=1}^{\infty} \frac{1}{n!} \cdot \left(\frac{i\hbar}{2}\right)^n P_n(f, g) \tag{17}$$

(there is only a finite number of nonzero terms in this series).

Proof. This result was independently announced in [11] without details about the proof. Our proof requires the following lemmas:

LEMMA 16. *Let f and g be two tempered distributions of $\mathscr{S}'(E)$ such that $\hat{f}$ and $\hat{g}$ have compact supports, then $f \circ g$ is defined and satisfies formula (17), the corresponding series converging in $\mathscr{S}'(E)$.*

This lemma was established in [10]. Let us recall its proof. The distributions $\hat{f}$ and $\hat{g}$ having compact supports, we have:

$$\langle \hat{f} \underset{\hbar/2}{*} \hat{g}, \varphi \rangle = \left\langle \hat{f}(x) \otimes \hat{g}(y), \exp\left(-i \frac{\hbar}{2} \omega(x, y)\right) \varphi(x + y) \right\rangle .$$

Let us consider the series expansion of the form:

$$\exp\left(-i \frac{\hbar}{2} \omega(x, y)\right) \varphi(x + y) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-i \frac{\hbar}{2} \omega(x, y)\right)^n \varphi(x + y)$$

and let us note that this series as well as its partial derivatives of all order converge uniformly on each compact subset of E for each function φ of $\xi(E)$. This is in fact immediately deduced from the properties of power series. It follows that the preceding series is convergent in the space $\xi(E \times E)$. So, we obtain the weak convergence in $\xi'(E)$ and then the weak convergence and therefore the strong convergence in $\mathscr{S}'(E)$ of the series:

$$\hat{f} \underset{\hbar/2}{*} \hat{g} = \hat{f} * \hat{g} + \sum_{n=1}^{\infty} \frac{1}{n!} \left(-i \frac{\hbar}{2}\right)^n \omega_{j_1 k_1} \ldots \omega_{j_n k_n} \cdot (x_{j_1} \ldots x_{j_n} \hat{f}(x)) * (y_{k_1} \ldots y_{k_n} \hat{g}(y))$$

where $*$ denotes the usual convolution product. Lemma 16 is then obtained by applying the symplectic Fourier transformation with the help of the formulas: $\mathscr{F}(x_j f) = i \sum_k \Lambda_{jk} \dfrac{\partial \hat{f}}{\partial x_k}$ and $\mathscr{F}(f \cdot g) = \hat{f} * \hat{g}$ (this last formula still holds for the symplectic Fourier transformation) and by using formula (16).

LEMMA 17. *If S is a tempered distribution of* $\mathscr{S}'(E)$, *if f is a function of* $\mathscr{O}_M(E)$ *and if* φ *is a function of* $\mathscr{S}(E)$, *then we have:*

$$\langle P_n(S,f),\varphi\rangle = \langle S, P_n(f,\varphi)\rangle$$

$$\langle P_n(f,S),\varphi\rangle = \langle S, P_n(\varphi,f)\rangle.$$

The proof of this lemma is obvious.

Let us return to the proof of proposition 15 and let us suppose for example that g is a polynomial, then $\hat{g}$ having a compact support, belongs to $\mathscr{O}'_\lambda(E)$, $\forall\lambda\in\mathbb{R}$ and also belongs to $\mathscr{O}'_C(E)$. (Let us note that $\mathscr{O}'_C(E) = \mathscr{O}'_o(E)$). Then $f\circ g$ is defined and by applying lemma 16, we obtain for every function φ of $\mathscr{S}(E)$ such that $\hat{\varphi}\in\mathscr{D}(E)$:

$$\langle f\circ g,\varphi\rangle = \langle \mathscr{F}(\hat{f} \underset{\hbar/2}{*} \hat{g}),\varphi\rangle = \langle \hat{f}, (\hat{g} \underset{\hbar/2}{*} \hat{\varphi})\check{}\,\rangle =$$

$$= \langle f, g\circ\varphi\rangle = \left\langle f, g\varphi + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n P_n(g,\varphi)\right\rangle .$$

Then, according to lemma 17, we see that the both hand sides of formula (17) are equal on a dense subset of $\mathscr{S}(E)$ and therefore are equal. (Let us recall that there is only a finite number of nonzero terms in series (17)). ■

CONCLUSION

Let us recall the main results of this paper

$S\in\mathscr{S}'(E)$	$\hat{S}\in\mathscr{S}'(E)$	$\mathrm{op}(S)\in\mathscr{L}(\mathbb{H}_\infty,\mathbb{H}'_\infty)$
Star product $S\circ T$	Twisted convolution product $\hat{S} \underset{\hbar/2}{*} \hat{T}$	product of operators $\mathrm{op}(S)\cdot\mathrm{op}(T)$
Admissible symbols $\{S\in\mathscr{S}'(E), S \underset{2/\hbar}{*} \mathscr{S}(E)\subset L^2(E)\}$	$\{\hat{S}\in\mathscr{S}'(E), \hat{S} \underset{\hbar/2}{*} \mathscr{S}(E)\subset L^2(E)\}$	$\mathscr{L}(\mathbb{H}_\infty,\mathbb{H})$
$\mathscr{O}'_{2/\hbar}(E)$	$\mathscr{O}'_{\hbar/2}(E)$	$\mathscr{L}(\mathbb{H}_\infty,\mathbb{H}_\infty)$
bounded $\frac{2}{\hbar}$-convolution operators of $L^2(E)$	bounded $\frac{\hbar}{2}$-convolution operators of $L^2(E)$	bounded operators of $\mathbb{H}$

ACKNOWLEDGEMENT

The author thanks Professor M. Flato for numerous suggestions and D. Sternheimer for helpful advice.

REFERENCES

[1] F. BAYEN, M. FLATO, C. FRONSDAL, A. LICHNEROWICZ and D. STERNHEIMER, *Deformation theory and quantization, I. Deformations of symplectic structures, II. Physical applications*, Ann. Phys., **111**, (1978), p. 61 - 110 and p. 111 - 151.

[2] A. VOROS, *Développements semi-classiques*, Thèse Orsay, n° 1843, (1977).

[3] A. VOROS, *An algebra of pseudo-differential operators and the asymptotics of quantum mechanics*, J. Functional Analysis, **29**, (1978), p. 104 - 132.

[4] D. KASTLER, *The C*-Algebras of a free Boson field*, Commun. Math. Phys. **1**, (1965), p. 14 - 48.

[5] G. LOUPIAS and S. MIRACLE-SOLE, *C*-Algèbres des systèmes canoniques II.*, Ann. Inst. Henri Poincaré, **6**, (1967), p. 39 - 58.

[6] M.A. ANTONETS, *The classical limit for Weyl quantization*, Lett. Math. Phys., **2**, (1978), p. 241 - 245.

[7] R. GOODMAN, *Analytic and entire vectors for representations of Lie groups*, Trans. Amer. Math. Soc., **143**, (1969), p. 55 - 76.

[8] J.M. MAILLARD, *Sur le produit de convolution gauche et la transformée de Weyl des distributions tempérées*, C.R. Acad. Sci. Paris, série I, **298**, (1984), p. 35 - 38 and Structure des convoluteurs à gauche de ($\mathbb{R}^{2l}$) et des symboles admissibles de la transformation de Weyl, C.R. Acad. Sci. Paris, série I, **301**, (1985), p. 719 - 721.

[9] F. BAYEN and J.M. MAILLARD, *Star exponentials of the elements of the inhomogeneous symplectic Lie algebra*, Lett. Math. Phys., **6**, (1982), p. 491 - 497.

[10] J.M. MAILLARD, *Sur la transformation de Weyl et le produit star des distributions tempérées*, preprint, Université de Dijon, (1982).

[11] J.B. KAMMERER, *L'algèbre des opérateurs de multiplication du star produit de* $\mathbb{R}^{2l}$, C.R. Acad. Sci. Paris, série I, **298**, (1984), p. 59 - 62.

[12] L. SCHWARTZ, *Théorie des distributions*, Hermann, Paris, (1966).

[13] F. TREVES, *Topological vector spaces, distributions and kernels*, Academic Press, New York, London, (1967).

[14] S. LANG, *Algebra*, Addison Wesley publishing company, (1969).

Manuscript received: March 10, 1986

Paper presented by
M. Flato